Please send mistakes or typos to xxchai@math.cuhk.edu.hk

1. Start with $\cos(A-B)=\cos A\cos B+\sin A\sin B$, $\sin(A+B)=\text{sinA}\cos B+\cos A\sin B$ and other elementry formulas, fill the blanks of 1a-1c and solve 1d.

1a. $\cos\alpha+\cos\beta=2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$

Proof Let $A=\frac{\alpha+\beta}{2}$ and $B=\frac{\alpha-\beta}{2}$. Then $A+B=\alpha$ and $A-B=\beta$, since

$$\begin{aligned}\cos(A-B)&=\cos A\cos B+\sin A\sin B\\ \cos(A+B)&=\cos A\cos B-\sin A\sin B,\end{aligned}$$

so

$$\begin{aligned}\cos\alpha+\cos\beta=&\cos(A+B)+\cos(A-B)\\ =&[\cos A\cos B-\sin A\sin B]+\\ &+[\cos A\cos B+\sin A\sin B]\\ =&2\cos A\cos B\\ =&2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.\end{aligned}$$

1b. $\sin\alpha+\sin\beta=2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

Proof Let $A=\frac{\alpha+\beta}{2}$ and $B=\frac{\alpha-\beta}{2}$. Then $A+B=\alpha$ and $A-B=\beta$, since

$$\begin{aligned}\sin(A-B)&=\text{sinA}\cos B-\cos A\sin B\\ \sin(A+B)&=\text{sinA}\cos B+\cos A\sin B\end{aligned},$$

we have

$$\begin{aligned}\sin\alpha+\sin\beta=&\sin(A+B)+\sin(A-B)\\ =&[\sin A\cos B+\cos A\sin B]+\\ &[\sin A\cos B-\cos A\sin B]\\ =&2\sin A\cos B\\ =&2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.\end{aligned}$$

1c. $\frac{1}{2}\cos\alpha+\frac{\sqrt{3}}{2}\sin\alpha=\_?$.

Proof $\frac{1}{2}\cos\alpha+\frac{\sqrt{3}}{2}\sin\alpha=\sin\frac{\pi}{6}\cos\alpha+\cos\frac{\pi}{6}\sin\alpha=\sin(\frac{\pi}{6}+\alpha)$.

1d. We introduce *complex numbers* to facilitate the calculation of trignometric functions. i is called the square root of -1, i.e. $i^2=-1$. Use $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2}$ to reduce the product $\cos x\cos 2x\cos 3x$ to summations of trignometric functions. Can you use $1a$ to do that?

Proof Use $\cos x=\frac{e^{ix}+e^{-ix}}{2}$,

$$\begin{aligned}\cos x\cos 2x\cos 3x &= \frac{1}{8}(e^{ix}+e^{-ix})(e^{2ix}+e^{-2ix})(e^{3ix}+e^{-3ix})\\ &= \frac{1}{8}(e^{3ix}+e^{ix}+e^{-ix}+e^{-3ix})(e^{3ix}+e^{-3ix})\\ &= \frac{1}{8}(e^{6ix}+e^{4ix}+e^{2ix}+1+1+e^{-2ix}+e^{-4ix}+e^{-6ix})\\ &= \frac{1}{8}(e^{6ix}+e^{-6ix})+\frac{1}{8}(e^{2ix}+e^{-2ix})+\frac{1}{8}(e^{4ix}+e^{-4ix})+\frac{1}{4}\\ &= \frac{1}{4}(\cos 6x+\cos 4x+\cos 2x+1).\end{aligned}$$

Or use 1a, we have

$$\begin{aligned}\cos x\cos 2x\cos 3x &= (\cos x\cos 2x)\cos 3x\\ &= \frac{1}{2}(\cos(x+2x)+\cos(2x-x))\cos 3x\\ &= \frac{1}{2}\cos 3x\cos 3x+\frac{1}{2}\cos x\cos 3x\\ &= \frac{1}{4}(1+\cos 6x)+\frac{1}{2}\cdot\frac{1}{2}(\cos(3x-x)+\cos(3x+x))\\ &= \frac{1}{4}(\cos 6x+\cos 4x+\cos 2x+1).\end{aligned}$$

2. Use *mathematical induction* to prove the Bernoulli inequality. If $x>-1$ and $x\neq 0$, then for all positive integer $n\geqslant 2$ that

$$(1+x)^n>1+nx.$$

Use the binomial theorem to get a partial proof only for the case $x>0$.

Proof Base case $n=2$. Since $x\neq 0$,

$$1+2x+x^2=(1+x)^2>1+2x.$$

Assume that $n=k$ that $(1+x)^n>1+nx$ holds, since $x+1>0$, we have

$$\begin{aligned}(1+x)^{k+1}=(1+x)^k(1+x) &> (1+kx)(1+x)\\ &= 1+kx+x+kx^2\\ &> 1+(k+1)x.\end{aligned}$$

So we have when $n=k+1$, $(1+x)^n>1+nx$ is true. By principle of mathematical induction, $(1+x)^n>1+nx$ is true for all positive integers $n\geqslant 2$.

When $x>0$, there is an quick proof:

$$(1+x)^n=\sum_{k=0}^{n} C_k^n x^k 1^{n-k}=\sum_{k=0}^{n} C_k^n x^k > C_0^n x^0 + C_1^n x^1 = 1+nx.$$

3a. $\lim_{n\to+\infty}(\sqrt{n}-\sqrt{n+1})=?$

Solution

$$\begin{aligned}\lim_{n\to\infty}(\sqrt{n}-\sqrt{n+1}) &= \lim_{n\to\infty}\frac{\sqrt{n}-\sqrt{n+1}}{1}\\ &= \lim_{n\to\infty}\frac{(\sqrt{n}-\sqrt{n+1})(\sqrt{n}+\sqrt{n+1})}{\sqrt{n}+\sqrt{n+1}}\\ &= \lim_{n\to\infty}\frac{(\sqrt{n})^2-(\sqrt{n+1})^2}{\sqrt{n}+\sqrt{n+1}}\\ &= \lim_{n\to\infty}\frac{-1}{\sqrt{n}+\sqrt{n+1}}\\ &=0.\end{aligned}$$

3b. **Use the 3a** to find $\lim_{n\to+\infty}(n-\sqrt{n^2+2n})=?$

Solution 3a tells us that

$$\lim_{n\to+\infty}(\sqrt{n^2+2n}-\sqrt{n^2+2n+1})=0.$$

So

$$\begin{aligned}&\lim_{n\to+\infty}(n-\sqrt{n^2+2n})\\ =&\lim_{n\to+\infty}[(n-\sqrt{n^2+2n+1})+(\sqrt{n^2+2n+1}-\sqrt{n^2+2n})]\\ =&\lim_{n\to+\infty}(n-\sqrt{n^2+2n+1})+\lim_{n\to+\infty}(\sqrt{n^2+2n+1}-\sqrt{n^2+2n})\\ =&\lim_{n\to+\infty}(n-(n+1))+0\\ =&-1.\end{aligned}$$

3c. Determine the monotonicity of sequences $\{a_n\}$ given by

$$a_n=\frac{1}{1+n}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}.$$

Does $\lim_{n\to\infty} a_n$ exists? (hint: $\frac{1}{2n} \leqslant \frac{1}{n+k} \leqslant \frac{1}{n}$) Use this limit and by contradiction to prove that

$$\lim_{n\to\infty} \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right)$$

does not exist.

<u>Proof</u> We calculate the difference

$$\begin{aligned}
a_{n+1}-a_n =& \left(\frac{1}{1+(n+1)}+\frac{1}{2+(n+1)}+\cdots+\frac{1}{(n+1)+(n+1)}\right)- \\
& \left(\frac{1}{1+n}+\frac{1}{2+n}+\cdots+\frac{1}{2n}\right) \\
=& \left(\frac{1}{2+n}+\frac{1}{3+n}+\cdots+\frac{1}{2(n+1)}\right)-\left(\frac{1}{1+n}+\frac{1}{2+n}+\cdots+\frac{1}{2n}\right) \\
=& \frac{1}{2(n+1)}+\frac{1}{2n+1}-\frac{1}{n+1} \\
=& \frac{1}{2n+1}-\frac{1}{2(n+1)}>0.
\end{aligned}$$

So the sequence a_n is increasing. Since

$$\frac{1}{2}=\frac{1}{2n}+\cdots+\frac{1}{2n} \leqslant a_n=\frac{1}{1+n}+\frac{1}{n+2}+\cdots+\frac{1}{n+n} \leqslant \frac{1}{n}+\cdots+\frac{1}{n}=1.$$

So by monotone convergence theorem, $\lim_{n\to\infty} a_n$ exists. Let $A=\lim_{n\to\infty} a_n$, then $\frac{1}{2} \leqslant A \leqslant 1$.

Let $S_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}+\frac{1}{n}$, we suppose that $\lim_{n\to\infty} S_n$ exists. Then $\lim_{n\to\infty} S_{2n}$ exists as well and

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty} S_{2n}.$$

But because $a_n=S_{2n}-S_n$,

$$0<A=\lim_{n\to\infty} a_n=\lim_{n\to\infty} (S_{2n}-S_n)=\lim_{n\to\infty} S_{2n}-\lim_{n\to\infty} S_n=0.$$

A contradiction. So the limit $\lim_{n\to\infty} S_n$ does not exist.

3d. (relatively hard) The factorial $n!$ of n is defined to be $n!=1\cdot 2\cdot 3\cdot \cdots \cdot n$. For $1 \leqslant k \leqslant n$, show that $(n+1-k)k \geqslant n$. Hence deduce that $(n!)^2 \geqslant n^n$. Use this to find that $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$.

<u>solution</u>

$$\begin{aligned}
(n+1-k)k-n =& nk-n+(1-k)k \\
=& n(k-1)+(1-k)k \\
=& (n-k)(k-1) \geqslant 0,
\end{aligned}$$

so for $1 \leqslant k \leqslant n$, $(n+1-k)k \geqslant n$. We list the inequalities

$$\begin{aligned} 1\cdot n = (n+1-1)1 &\geqslant n \\ 2\cdot(n-1)(n+1-2)2 &\geqslant n \\ \vdots \ &\vdots \\ k\cdot(n+1-k) = (n+1-k)k &\geqslant n \\ \vdots \ &\vdots \\ n\cdot 1 = (n+1-n)n &\geqslant n. \end{aligned}$$

Taking products, we have

$$(1\cdot n)(2\cdot(n-1))\cdots(n\cdot 1) = n!n! \geqslant n^n.$$

So $\frac{1}{(n!)^2} \leqslant \frac{1}{n^n} \Rightarrow \frac{1}{n!} \leqslant n^{-\frac{1}{2}n} \Rightarrow \frac{1}{\sqrt[n]{n!}} \leqslant n^{-\frac{1}{2}}$ which gives

$$0 \leqslant \frac{1}{\sqrt[n]{n!}} \leqslant n^{-\frac{1}{2}}.$$

By squeeze theorem,

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0.$$

4. **Limits and functions**

4a. $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+\sin x} = \_?$ and $\lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{x+\sin x} = \_?$

solution

$$\begin{aligned} \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+\sin x} &= \lim_{y\to+\infty} \frac{\sqrt{y^2+1}}{-y+\sin(-y)} \\ &= \lim_{y\to+\infty} \frac{\sqrt{y^2+1}\,/\,y}{[-y+\sin(-y)]\,/\,y} \\ &= \lim_{y\to+\infty} \frac{\sqrt{1+(\frac{1}{y})^2}}{-1+\frac{\sin(-y)}{y}} \\ &= \frac{1}{-1+0} = -1. \end{aligned}$$

Similarly, $\lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{x+\sin x} = 1$.

4b. $\lim_{x\to 1} \frac{x+x^2+\cdots+x^n-n}{x-1} = \_?$

solution For all $n \geqslant 1$, we have

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Then

$$\begin{aligned}
\lim_{x\to 1} \frac{x + x^2 + \cdots + x^n - n}{x - 1} &= \lim_{x\to 1} \frac{(x-1) + (x^2 - 1) + \cdots + (x^n - 1)}{x - 1} \\
&= \lim_{x\to 1} [1 + (1 + x) + \cdots + (1 + x + x^2 + \cdots + x^{n-1})] \\
&= 1 + 2 + 3 + \cdots + n \\
&= \frac{n(n+1)}{2}.
\end{aligned}$$

4c. It is know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Find

$$\lim_{x\to 0} \frac{\cos(3\tan x) - \cos(\tan x)}{x^2}$$

and

$$\lim_{x\to 0} \frac{1 - \cos x \cos 2x \cos 3x}{1 - \cos x}.$$

solution we use $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to obtain

$$\begin{aligned}
\lim_{x\to 0} \frac{1 - \cos x}{x^2} &= \lim_{x\to 0} \frac{(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}) - (\cos^2\frac{x}{2} - \sin^2\frac{x}{2})}{x^2} \\
&= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{4 \cdot (\frac{x}{2})^2} = \frac{1}{2}\lim_{x\to 0}(\frac{\sin\frac{x}{2}}{\frac{x}{2}}) = \frac{1}{2}.
\end{aligned}$$

So

$$\begin{aligned}
&\lim_{x\to 0} \frac{\cos(3\tan x) - \cos(\tan x)}{x^2} \\
&= \lim_{x\to 0} [\frac{1 - \cos(\tan x)}{\tan^2 x} - \frac{9(1 - \cos(\tan 3x))}{(3\tan x)^2}] \frac{\sin^2 x / (\cos^2 x)}{x^2} \\
&= \lim_{x\to 0} \frac{1 - \cos(\tan x)}{\tan^2 x} - 9\lim_{x\to 0} \frac{1 - \cos(\tan 3x)}{(3\tan x)^2} \\
&= \frac{1}{2} - 9 \cdot \frac{1}{2} = -4.
\end{aligned}$$

We already proved that

$$\cos x \cos 2x \cos 3x = \frac{1}{4}(\cos 6x + \cos 4x + \cos 2x + 1).$$

So

$$
\begin{aligned}
&\lim_{x\to0}\frac{1-\cos x\cos 2x\cos3x}{1-\cos x}\\
=&\lim_{x\to0}\frac{1-\frac{1}{4}(\cos6x+\cos4x+\cos2x+1)}{1-\cos x}\\
=&\frac{1}{4}\lim_{x\to0}\frac{1-\cos6x}{1-\cos x}+\frac{1}{4}\lim_{x\to0}\frac{1-\cos4x}{1-\cos x}+\frac{1}{4}\lim_{x\to0}\frac{1-\cos 2x}{1-\cos x}\\
=&\frac{1}{4}\lim_{x\to0}\frac{1-\cos 6x}{(6x)^2}\cdot\frac{(6x)^2}{x^2}\cdot\frac{x^2}{1-\cos x}\\
&+\frac{1}{4}\lim_{x\to0}\frac{1-\cos 4x}{(4x)^2}\cdot\frac{(4x)^2}{x^2}\cdot\frac{x^2}{1-\cos x}\\
&+\frac{1}{4}\lim_{x\to0}\frac{1-\cos 2x}{(2x)^2}\cdot\frac{(2x)^2}{x^2}\cdot\frac{x^2}{1-\cos x}\\
=&\frac{1}{4}\cdot\frac{1}{2}\cdot6^2\cdot2+\frac{1}{4}\cdot\frac{1}{2}\cdot4^2\cdot2+\frac{1}{4}\cdot\frac{1}{2}\cdot2^2\cdot2=14.
\end{aligned}
$$

4d. $\lim_{x\to+\infty}\frac{\ln(x^3+2x+3)}{\ln(x^8+2018x+1)}=\_?$

<u>solution</u>

$$
\begin{aligned}
&\lim_{x\to+\infty}\frac{\ln(x^3+2x+3)}{\ln(x^8+2018x+1)}\\
=&\lim_{x\to+\infty}\frac{\ln(x^3+2x+3)-\ln(x^3)+\ln(x^3)}{\ln(x^8+2018x+1)-\ln(x^8)+\ln(x^8)}\\
=&\lim_{x\to+\infty}\frac{3\ln x+\ln\frac{x^3+2x+3}{x^3}}{8\ln x+\ln\frac{x^8+2018x+1}{x^8}}\\
=&\lim_{x\to+\infty}\frac{3+\ln(1+2x^{-2}+3x^{-3})/\ln x}{8+\ln(1+2018x^{-7}+x^{-8})/\ln x}\\
=&\frac{3}{8}.
\end{aligned}
$$